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The normal force exerted by creeping flow on a small sphere touching a plane

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The hydrodynamic force experienced by a small solid sphere of radius a_p resting on a solid plane wall in axisymmetric stagnation flow, $\mathbf{v}_\infty = \Omega(-z^2\mathbf{i}_z + z\tilde{\omega}\mathbf{i}_{\tilde{\omega}})$, or in planar stagnation flow, $\mathbf{v}_\infty = \Omega(-z^2\mathbf{i}_z + 2zx\mathbf{i}_x)$, is computed on the basis of Stokes' creeping flow equations. In both cases, as well as for any flow whose z component of velocity is $-\Omega z^2$, this force is found to be $F_z = -60{\cdot}87\mu\Omega a_p^3$, where μ is the viscosity of the fluid. The uniform flow parallel to the line of centres of two touching spheres of arbitrary radii is also solved.

Introduction

In the theory of particle capture by filtration or scrubbing one must know the hydrodynamic forces exerted on particles suspended in the flow near collectors. In many practical cases the suspended particles are much smaller than the collectors; it is then a good approximation to assume that the flow field past the collectors is undisturbed by the particles except in the latters' immediate vicinity and that at separations greater than several particle diameters particle centres move along undisturbed fluid streamlines. Within several particle diameters, however, hydrodynamic interactions become increasingly important and the approximation that particle centres move along undisturbed fluid streamlines is no longer valid. In deriving an expression for the efficiency of particle capture by hydrodynamic and London–van der Waals forces (Spielman & Goren 1970) it is most important to estimate the hydrodynamic force of attraction for very small gaps. In this article we calculate the limiting hydrodynamic force attained when the particle and collector touch and the flow at infinity is along their line of centres. The particle is assumed to be so small compared to the collector that the latter can be treated as a plane except in so far as its size and shape determine the flow field far from the particle. For spherical collectors the flow far from the particle is taken to be axisymmetric stagnation flow,

$$\mathbf{v}_\infty = \Omega(-z^2\mathbf{i}_z + z\tilde{\omega}\mathbf{i}_{\tilde{\omega}}),$$

whereas for cylindrical collectors the flow is planar stagnation flow,

$$\mathbf{v}_\infty = \Omega(-z^2\mathbf{i}_z + 2zx\mathbf{i}_x).$$

The parameter Ω is determined by the flow model, i.e. whether the collector can be considered isolated or whether the flow past a given collector is influenced by neighbouring collectors. For example, for cylindrical collectors of radius a_F,

$\Omega = 2A_F U a_F^{-2}$, where U is the uniform velocity at infinity parallel to the line of centres; using Lamb's solution for isolated cylinders, $A_F = \frac{1}{2}\{2 - \ln(2\rho a_F U/\mu)\}^{-1}$ whereas in fibrous mats of porosity ϵ, $A_F \approx \frac{1}{4}(1-\epsilon)^{-\frac{1}{2}}$. For spherical collectors of radius a_s, $\Omega = \frac{3}{2}A_s U a_s^{-2}$; using Stokes' solution for isolated spheres, $A_s = 1$.

The creeping motion past two spheres of arbitrary sizes and spacing has been solved exactly by Stimson & Jeffery (1926) using bispherical co-ordinates. The stream function is given as an infinite series involving Gegenbauer polynomials. This series breaks down as the spheres touch, but it should be possible to determine the force on a given particle by an appropriate limiting procedure. Unfortunately, the expressions given by Stimson & Jeffery are very cumbersome, except for the special case of equal spheres, and so it was thought better to approach the problem afresh using tangent-sphere co-ordinates. Similarly, the creeping flow past two touching spheres of arbitrary size can be solved exactly but the equations are again somewhat cumbersome. Accordingly we make the approximation at the outset that the collector is so large compared to the particle that it can be treated as a solid plane wall.

Another problem relevant to the capture of small particles by much larger collectors concerns the force and torque experienced by a sphere touching a plane solid wall in uniform shear flow. This has recently been treated exactly by O'Neill (1968) using tangent-sphere co-ordinates and by Goldman, Cox & Brenner (1967) who extrapolated results obtained with bispherical co-ordinates to zero gap thickness.

Sphere in axisymmetric stagnation flow

The flow field far from the sphere is taken to be axisymmetric stagnation flow:

$$\left.\begin{aligned} \mathbf{v}_\infty &= \Omega\{-z^2\mathbf{i}_z + z\tilde{\omega}\mathbf{i}_{\tilde{\omega}}\}, \\ \psi_\infty &= \tfrac{1}{2}\Omega z^2\tilde{\omega}^2. \end{aligned}\right\} \tag{1}$$

It is convenient to adopt tangent-sphere co-ordinates (ξ, η, θ) which are related to cylindrical co-ordinates $(z, \tilde{\omega}, \theta)$ by the equations

$$\left.\begin{aligned} z &= \xi(\eta^2+\xi^2)^{-1}, \\ \tilde{\omega} &= \eta(\eta^2+\xi^2)^{-1}, \\ \theta &= \theta. \end{aligned}\right\} \tag{2}$$

The surface ξ = constant is a sphere of radius $|2\xi|^{-1}$ centred at $z = \frac{1}{2}\xi^{-1}$ and thus touches the x, y plane at the origin; $\xi = 0$ is the x, y plane. The surface η = constant is a torus formed by rotating a circle of radius $\frac{1}{2}\eta^{-1}$ centred at $\tilde{\omega} = \frac{1}{2}\eta^{-1}$ about the z axis; $\eta = 0$ is the z axis. The surface θ = constant is a plane containing the z axis. (ξ, η, θ) are orthogonal curvilinear co-ordinates with scale factors

$$h_\xi = h_\eta = h = (\eta^2+\xi^2). \tag{3}$$

Note that $r^2 = \tilde{\omega}^2 + z^2 = (\eta^2+\xi^2)^{-1}$ so that infinitely far from the particle both η and ξ simultaneously approach zero.

In terms of the new co-ordinates, the stream function at infinity is

$$\psi_\infty = \tfrac{1}{2}\Omega\eta^2\xi^2(\eta^2+\xi^2)^{-4}. \tag{4}$$

The stream function ψ is written as

$$\psi(\eta,\xi) = \psi_\infty(\eta,\xi) + \hat{\psi}(\eta,\xi) \tag{5}$$

and must satisfy the equation of creeping motion for axisymmetric flow (Happel & Brenner 1965)

$$E^2E^2\psi = 0, \tag{6}$$

where
$$E^2 = \tilde{\omega}\frac{\partial}{\partial\tilde{\omega}}\frac{1}{\tilde{\omega}}\frac{\partial}{\partial\tilde{\omega}} + \frac{\partial^2}{\partial z^2} = \eta(\eta^2+\xi^2)\left\{\frac{\partial}{\partial\eta}\frac{\eta^2+\xi^2}{\eta}\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\xi}\frac{\eta^2+\xi^2}{\eta}\frac{\partial}{\partial\xi}\right\}. \tag{7}$$

Since ψ_∞ itself satisfies (6) the equation of motion becomes

$$E^2E^2\hat{\psi} = 0. \tag{8}$$

The boundary conditions are no-slip on the solid plane $\xi = 0$ and the solid sphere $\xi = \kappa = (2a_p)^{-1}$:

$$\text{on}\quad \xi = 0,\ \psi = 0,\quad \therefore\ \hat{\psi} = -\psi_\infty = 0; \tag{9}$$

$$\text{on}\quad \xi = 0,\ \frac{\partial\psi}{\partial\xi} = 0;\quad \therefore\ \frac{\partial\hat{\psi}}{\partial\xi} = -\frac{\partial\psi_\infty}{\partial\xi} = 0; \tag{10}$$

$$\text{on}\quad \xi = \kappa,\quad \psi = 0,\quad \therefore\ \hat{\psi} = -\psi_\infty = -\tfrac{1}{2}\Omega\eta^2\kappa^2(\eta^2+\kappa^2)^{-4}; \tag{11}$$

$$\text{on}\quad \xi = \kappa,\quad \frac{\partial\psi}{\partial\xi} = 0,\quad \therefore\ \frac{\partial\hat{\psi}}{\partial\xi} = -\frac{\partial\psi_\infty}{\partial\xi} = \frac{4\Omega\eta^2\kappa^3}{(\eta^2+\kappa^2)^5} - \frac{\Omega\eta^2\kappa}{(\eta^2+\kappa^2)^4}. \tag{12}$$

In order that the disturbance due to the presence of the sphere vanish at infinity we also require

$$\hat{\psi}(0,0) = 0. \tag{13}$$

Since $h^{-2}\tilde{\omega}^{-2} = \eta^2$ is of the form $P(\eta)+Q(\xi)$, $E^2f = 0$ has a solution of the form $f = \tilde{\omega}^{-\frac{1}{2}}U(\eta)\,V(\xi)$ (see Happel & Brenner 1965). Substitution into (7) shows f is of the form

$$f = (\eta^2+\xi^2)^{-\frac{1}{2}}\int_0^\infty \lambda\eta J_1(\lambda\eta)\{A^*(\lambda)\cosh\lambda\xi + B^*(\lambda)\sinh\lambda\xi\}\,d\lambda, \tag{14}$$

where J_1 is the Bessel function of the first kind and order unity. In order that the velocity remain finite along the z axis a similar term involving $Y_1(\lambda\eta)$ has been omitted. It is easy to show that if f satisfies $E^2F = 0$, then $E^2E^2(zf) = 0$. Thus the solution to (8) is a term of the form (14) plus $\xi(\eta^2+\xi^2)^{-1}$ times a similar term. Using the differential equation satisfied by $J_1(\lambda\eta)$ and two integrations by parts, we may write

$$(\eta^2+\xi^2)^{\frac{3}{2}}\,\hat{\psi} = \int_0^\infty \lambda\eta J_1(\lambda\eta)\,\mathscr{F}(\xi)\,d\lambda, \tag{15}$$

where

$$\mathscr{F}(\xi) = A(\lambda)\cosh\lambda\xi + B(\lambda)\sinh\lambda\xi + C(\lambda)\,\lambda\xi\cosh\lambda\xi + D(\lambda)\,\lambda\xi\sinh\lambda\xi. \tag{16}$$

Application of (9) and 10) yields

$$\mathscr{F}(0) = 0,\quad \therefore\ A = 0; \tag{17}$$

$$\mathscr{F}'(0) = 0,\quad \therefore\ B = -C. \tag{18}$$

Application of (11) and (12) yields

$$-\tfrac{1}{2}\eta\Omega^2\kappa^2(\eta^2+\kappa^2)^{-\frac{5}{2}} = \int_0^\infty \lambda\eta J_1(\lambda\eta)\mathscr{F}(\kappa)\,d\lambda, \tag{19}$$

$$\frac{5\Omega\eta^2\kappa^3}{2(\eta^2+\kappa^2)^{\frac{7}{2}}}-\frac{\Omega\eta^2\kappa}{(\eta^2+\kappa^2)^{\frac{5}{2}}} = \int_0^\infty \lambda\eta J_1(\lambda\eta)\mathscr{F}'(\kappa)\,d\lambda. \tag{20}$$

Inverting these Hankel transforms with the aid of formulas given in Edelyi *et al.* (1954, chapter VIII) gives

$$\mathscr{F}(\kappa) = \int_0^\infty \eta J_1(\lambda\eta)\{-\tfrac{1}{2}\Omega\eta\kappa^2(\eta^2+\kappa^2)^{-\frac{5}{2}}\}\,d\eta = -\tfrac{1}{6}\Omega\lambda\kappa e^{-\lambda\kappa} \tag{21}$$

or

$$C\{\lambda\kappa\cosh\lambda\kappa-\sinh\lambda\kappa\}+D\lambda\kappa\sinh\lambda\kappa = -\tfrac{1}{6}\Omega\lambda\kappa e^{-\lambda\kappa} \tag{22}$$

and

$$\kappa\mathscr{F}'(\kappa) = \int_0^\infty \eta J_1(\lambda\eta)\left\{\frac{5\Omega\eta\kappa^4}{2(\eta^2+\kappa^2)^{\frac{7}{2}}}-\frac{\Omega\eta\kappa^2}{(\eta^2+\kappa^2)^{\frac{5}{2}}}\right\}d\eta = \tfrac{1}{6}\Omega\lambda\kappa(\lambda\kappa-1)\,e^{-\lambda\kappa} \tag{23}$$

or

$$C\lambda^2\kappa^2\sinh\lambda\kappa+D\{\lambda\kappa\sinh\lambda\kappa+\lambda^2\kappa^2\cosh\lambda\kappa\} = \tfrac{1}{6}\Omega\lambda\kappa(\lambda\kappa-1)\,e^{-\lambda\kappa}. \tag{24}$$

Solving (22) and (24) for the functions C and D gives

$$C = \frac{\Omega\lambda^2\kappa^2}{6\{\sinh^2\lambda\kappa-\lambda^2\kappa^2\}} \tag{25}$$

and

$$D = -\frac{\Omega\{\lambda\kappa(\lambda\kappa-1)+e^{-\lambda\kappa}\sinh\lambda\kappa\}}{6\{\sinh^2\lambda\kappa-\lambda^2\kappa^2\}}. \tag{26}$$

Thus the stream function is completely specified.

To determine the net force in the z direction on an axisymmetric body we use the formula derived by Stimson & Jeffery (1926) and Happel & Brenner (1965),

$$F_z = \pi\mu\int \tilde{\omega}^3\frac{\partial}{\partial n}\left(\frac{E^2\psi}{\tilde{\omega}^2}\right)ds, \tag{27}$$

where n and s are the normal and tangential co-ordinates to the surface. Here $\partial/\partial n = h\,\partial/\partial\xi$ and $ds = h^{-1}d\eta$. It is tedious to apply this formula on the surface of the sphere $\xi = \kappa$. However, by a steady-state momentum balance on a 'large cylinder' of fluid bounded on the bottom by the plane wall and sphere it is possible to show from the behaviour of ψ for small ξ and η that the force acting on the sphere is equal to minus the net force due to the disturbance flow $\hat{\psi}$ acting on the solid plane $\xi = 0$. Thus

$$F_z = -\pi\mu\int_0^\infty\left[\tilde{\omega}^3\frac{\partial}{\partial\xi}\left(\frac{E^2\hat{\psi}}{\tilde{\omega}^2}\right)\right]_{\xi=0}d\eta. \tag{28}$$

Carrying out the indicated operations gives

$$\begin{aligned} F_z &= -\pi\mu\int_0^\infty\int_0^\infty \lambda\eta J_1(\lambda\eta)\mathscr{F}'''(0)\,d\lambda\,d\eta \\ &= -2\pi\mu\int_0^\infty \lambda^2 C(\lambda)\,d\lambda \\ &= -\mu\Omega a_p^3\frac{8\pi}{3}\int_0^\infty\frac{x^4\,dx}{\sinh^2 x-x^2}. \end{aligned} \tag{29}$$

The integral was evaluated numerically, the result being

$$F_z = -60{\cdot}87\mu\Omega a_p^3. \tag{30}$$

Sphere in planar stagnation flow

For a spherical particle captured by a much larger cylinder the flow field far from the particle is taken to be planar stagnation flow

$$\mathbf{v}_\infty = \Omega\{-z^2\mathbf{i}_z + 2zx\mathbf{i}_x\}. \tag{31}$$

As before, on the wall ($\xi = 0$) and sphere surface ($\xi = \kappa = (2a_p)^{-1}$) the fluid velocity vanishes. Because the flow field at infinity is not axisymmetric it is difficult to solve this three-dimensional problem. Nevertheless it is clear from the symmetry that the net force on the sphere will be in the z direction. Let this force be F_z.

The planar stagnation flow field

$$\mathbf{v}_\infty = \Omega\{-z^2\mathbf{i}_z + 2zy\mathbf{i}_y\}, \tag{32}$$

with no slip on the plane and sphere, must produce the same net force on the sphere F_z. Since the equation of motion and boundary conditions are linear, the sum of the solutions due to (31) and (32) will also be a solution to the equations of motion and will result in the force $2F_z$. For the sum, the flow field at infinity is

$$\mathbf{v}_\infty = 2\Omega\{-z^2\mathbf{i}_z + zx\mathbf{i}_x + zy\mathbf{i}_y\} = 2\Omega\{-z^2\mathbf{i}_z + z\tilde{\omega}\mathbf{i}_{\tilde{\omega}}\}. \tag{33}$$

With this flow at infinity and no slip on the plane and sphere we have exactly the same problem treated in the preceding section. Thus,

$$F_z = -60{\cdot}87\mu\Omega a_p^3. \tag{34}$$

In fact, if the undisturbed velocity far from the sphere is given by

$$u(x,y,z)\,\mathbf{i} + v(x,y,z)\,\mathbf{j} - \Omega z^2\mathbf{k}$$

(the last term being so determined to second order in x, y and z by the no-slip and continuity conditions) the normal component of the force is given correctly by (34) for arbitrary u and v. The reason for this is that to the creeping flow approximation the force caused by any flow reverses sign when the direction of flow is reversed, and since the normal component of the force must be independent of the direction of u or v these flows can contribute no net normal force. When the z component of velocity is other than $-\Omega z^2$, however, a different normal force results.

Uniform flow past two touching spheres of arbitrary sizes

For completeness, we give here the solution for fluid moving with uniform velocity U at infinity parallel to the line of centres of two touching spheres of arbitrary radii. Now the stream function at infinity is

$$\psi_\infty = \tfrac{1}{2}U\tilde{\omega}^2 = \tfrac{1}{2}U\eta^2(\eta^2+\xi^2)^{-2}. \tag{35}$$

Again

$$\psi = \psi_\infty + \hat{\psi}$$

with $\hat{\psi}$ given by (15) and (16). Applying the no-slip boundary conditions

$$\left.\begin{aligned} &\psi = 0 \quad \text{and} \quad \frac{\partial\psi}{\partial\xi} = 0 \quad \text{on} \quad \xi = \kappa = \frac{1}{2a_p} \\ \text{and} \quad &\psi = 0 \quad \text{and} \quad \frac{\partial\psi}{\partial\xi} = 0 \quad \text{on} \quad \xi = -\epsilon = -\frac{1}{2a_s} \end{aligned}\right\} \tag{36}$$

and inverting the resulting Hankel transforms gives the following four linear algebraic equations for the functions A, B, C and D:

$$A\cosh\lambda\kappa + B\sinh\lambda\kappa + C\lambda\kappa\cosh\lambda\kappa + D\lambda\kappa\sinh\lambda\kappa = -U(1+\lambda\kappa)\,e^{-\lambda\kappa}/2\lambda^2;$$

$$A\cosh\lambda\epsilon - B\sinh\lambda\epsilon - C\lambda\epsilon\cosh\lambda\epsilon + D\lambda\epsilon\sinh\lambda\epsilon = -U(1+\lambda\epsilon)\,e^{-\lambda\epsilon}/2\epsilon^2;$$

$$\begin{aligned}A\lambda\kappa\sinh\lambda\kappa + B\lambda\kappa\cosh\lambda\kappa + C[\lambda\kappa\cosh\lambda\kappa + \lambda^2\kappa^2\sinh\lambda\kappa]&\\ + D[\lambda\kappa\sinh\lambda\kappa + \lambda^2\kappa^2\cosh\lambda\kappa] &= \tfrac{1}{2}U\kappa^2e^{-\lambda\kappa};\end{aligned}$$

$$\begin{aligned}A\lambda\epsilon\sinh\lambda\epsilon - B\lambda\epsilon\cosh\lambda\epsilon - C[\lambda\epsilon\cosh\lambda\epsilon + \lambda^2\epsilon^2\sinh\lambda\epsilon]&\\ + D[\lambda\epsilon\sinh\lambda\epsilon + \lambda^2\epsilon^2\cosh\lambda\epsilon] &= \tfrac{1}{2}U\epsilon^2e^{-\lambda\epsilon}. \qquad (37)\end{aligned}$$

The forces on the two spheres may be computed with the aid of (27). Again it is more convenient to apply this formula to a large cylinder of fluid bounded on the bottom by the plane $\xi = 0$. After a great deal of manipulation, the result simplifies to

$$\left.\begin{aligned}F_z(\text{particle } a_p) &= 2\pi\mu\int_0^\infty \lambda^2(A+B)\,d\lambda,\\ \text{and}\qquad F_z(\text{particle } a_s) &= 2\pi\mu\int_0^\infty \lambda^2(A-B)\,d\lambda.\end{aligned}\right\} \qquad (38)$$

The force on the aggregate is

$$F_z(\text{aggregate}) = 4\pi\mu\int_0^\infty \lambda^2 A\,d\lambda, \qquad (39)$$

which may easily be seen to be in agreement with the result obtained from the relation

$$F_z(\text{aggregate}) = 8\pi\mu\lim_{r\to\infty}\left[\frac{r(\psi-\psi_\infty)}{\varpi^2}\right]. \qquad (40)$$

Equations (37) were solved for A and B and the forces on the two particles then computed for selected values of the parameter $R = a_p/a_s$ by numerically carrying out the integrations indicated in (38). The results may be expressed in the form

$$F_z(\text{particle } a_p) = 6\pi\mu U a_p\, f(R) \qquad (41)$$

with $f(R)$ given in the accompanying table.

R	$f(R)$	R	$f(R)$
1·00	0·645	1·0	0·645
0·50	0·365	2·0	0·866
0·25	0·155	4·0	0·965
0·10	0·0364	10·0	0·997
0·05	0·0105	20·0	1·000
$R \to 0$	$4{\cdot}844R^2$	∞	1·000

For $a_p \gg a_s$ the sphere of radius a_s has a very minor effect and the force on the sphere of radius a_p approaches that given by Stokes for an isolated sphere. For $a_p \ll a_s$, $f(R)$ may be estimated from the formula $f = 4{\cdot}844R^2$ obtained from the results for axisymmetric stagnation flow with $\Omega = \frac{3}{2}Ua_s^{-2}$. The numerical result for $R = 1$ is in excellent agreement with the calculations of Faxen and Dahl for equal spheres reported by Happel & Brenner (1965).

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